

SMOOTH STRUCTURES ON COMPLEX PROJECTIVE SPACES

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ABSTRACT. We compute the group $\mathcal{C}(\mathbb{CP}^m)$ of concordance classes of smoothings of complex projective space \mathbb{CP}^m for $m = 5, 6, 7, 8$. As an application, for $m = 7, 8$, we compute the smooth tangential homotopy structure set of \mathbb{CP}^m and also determine the number of smooth structures on \mathbb{CP}^m with fixed Pontryagin classes.

1. INTRODUCTION

The study of smooth homotopy complex projective spaces received its first impetus as a means of classifying free circle actions on homotopy spheres [7, 8, 14, 15]. D. Sullivan [20] later classified PL homotopy complex projective spaces as an application of his characteristic variety theorem. The surgery classification of smooth homotopy complex projective spaces was initiated and given a complete smooth classification of manifolds homotopy equivalent to \mathbb{CP}^m , where $m \leq 6$, by Brumfiel [3, 4]. In [9], for $m = 3$ and 4, it was proved that the group $\mathcal{C}(\mathbb{CP}^m)$ of concordance classes of smoothings of \mathbb{CP}^m is isomorphic to the group of smooth homotopy spheres Θ_{2m} [11].

In this paper we try to compute the following:

- Theorem A.** a) $\mathcal{C}(\mathbb{CP}^5) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2$.
 b) $\mathcal{C}(\mathbb{CP}^6) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2$.
 c) $\mathcal{C}(\mathbb{CP}^7) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
 d) $\mathcal{C}(\mathbb{CP}^8) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(cf. Theorem 2.4 and Theorem 2.6). Computations of the groups of concordance classes of smoothings are obviously of importance in the classification of smooth simply connected manifolds with a given homotopy type and the classification of diffeomorphism classes of smooth structures. In particular, Theorem A gives the number of smooth structures on \mathbb{CP}^m , where $m = 5, 6, 7, 8$.

In this paper, for $m = 7, 8$, we also compute the smooth tangential homotopy structure set of \mathbb{CP}^m by use of the tangential surgery exact sequence [6, 13] and prove the following.

- Theorem B.** a) *There are exactly four smooth structures on \mathbb{CP}^7 having the same Pontryagin classes of \mathbb{CP}^7 .*
 b) *There are exactly eight smooth structures on \mathbb{CP}^8 having the same Pontryagin classes of \mathbb{CP}^8 .*

(cf. Theorem 3.7).

Organisation of the paper: In section 2, we introduce some preliminaries from smoothing theory [3, 4] and prove a result relating this to a computation in stable cohomotopy. In section 3, we apply results of Section 2, by using the tangential surgery exact sequence ([13]) and Sullivan's and Crowley-Hambleton ([6]) identifications of the normal invariants,

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we prove results regarding the classification of smooth manifolds having the same Pontryagin classes of \mathbb{CP}^m .

Notation: Denote by $O = \text{colim}_n O(n)$, $Top = \text{colim}_n Top(n)$, $F = \text{colim}_n F(n)$ the direct limit of the groups of orthogonal transformations, homeomorphisms, and homotopy equivalences respectively. In this paper all manifolds will be closed smooth, oriented and connected, and all homeomorphisms and diffeomorphisms are assumed to preserve orientation, unless otherwise stated.

2. GROUP OF CONCORDANCE CLASSES OF SMOOTH STRUCTURES

We recall some terminology from [11]:

Definition 2.1. (a) A homotopy m -sphere Σ^m is an oriented smooth closed manifold homotopy equivalent to the standard unit sphere \mathbb{S}^m in \mathbb{R}^{m+1} .
 (b) A homotopy m -sphere Σ^m is said to be exotic if it is not diffeomorphic to \mathbb{S}^m .
 (c) Two homotopy m -spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists an orientation preserving diffeomorphism $f : \Sigma_1^m \rightarrow \Sigma_2^m$.

The set of equivalence classes of homotopy m -spheres is denoted by Θ_m . The equivalence class of Σ^m is denoted by $[\Sigma^m]$. When $m \geq 5$, Θ_m forms an abelian group with group operation given by connected sum $\#$ and the zero element represented by the equivalence class of \mathbb{S}^m . M. Kervaire and J. Milnor [11] showed that each Θ_m is a finite group; in particular, $\Theta_m \cong \mathbb{Z}_2$, where $m = 8, 14, 16$, and $\Theta_{10} \cong \mathbb{Z}_6$.

Definition 2.2. Let M be a topological manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a homeomorphism $f : N \rightarrow M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a diffeomorphism $g : N_1 \rightarrow N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1 , i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \rightarrow M \times [0, 1]$ such that $F|_{N_1 \times 0} = f_1$ and $F|_{N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

Start by noting that there is a homeomorphism $h : M^n \# \Sigma^n \rightarrow M^n$ ($n \geq 5$) which is the inclusion map outside of homotopy sphere Σ^n and well defined up to topological concordance. We will denote the class in $\mathcal{C}(M)$ of $(M^n \# \Sigma^n, h)$ by $[M^n \# \Sigma^n]$. (Note that $[M^n \# \mathbb{S}^n]$ is the class of (M^n, Id) .)

Definition 2.3. ($Cat = Diff$ or Top -structure sets) Let M be a closed Cat -manifold. We define the Cat -structure set $\mathcal{S}^{Cat}(M)$ to be the set of equivalence classes of pairs (N, f) where N is a closed Cat -manifold and $f : N \rightarrow M$ is a homotopy equivalence. And the equivalence relation is defined as follows:

$$(N_1, f_1) \sim (N_2, f_2) \text{ if there is a } Cat\text{-isomorphism } h : N_1 \rightarrow N_2 \text{ such that } f_2 \circ h \text{ is homotopic to } f_1.$$

Let $f_M : M^m \rightarrow S^m$ be a degree one map. Note that f_M is well-defined up to homotopy. Composition with f_M defines a homomorphism

$$f_M^* : [S^m, Top/O] \rightarrow [M^m, Top/O],$$

and in terms of the identifications

$$\Theta_m = [S^m, Top/O] \text{ and } \mathcal{C}(M^m) = [M^m, Top/O]$$

given by [12, p. 25 and 194], f_M^* becomes $[\Sigma^m] \mapsto [M^m \# \Sigma^m]$.

We start by recalling some facts from smoothing theory [3, 4]. There are H -spaces SF , F/O and Top/O and H -space maps $\phi : SF \rightarrow F/O$, $\psi : Top/O \rightarrow F/O$ such that

$$(2.1) \quad \phi_* : [\mathbb{CP}^m, SF] \rightarrow [\mathbb{CP}^m, F/O]$$

$$(2.2) \quad \psi_* : [\mathbb{CP}^m, Top/O] \rightarrow [\mathbb{CP}^m, F/O]$$

are monomorphism for all $m \geq 1$,

$$(2.3) \quad [\mathbb{CP}^{m-1}, F/O] = \mathbb{Z}^{[\frac{m-1}{2}]} \oplus [\mathbb{CP}^{m-1}, SF],$$

where

$$\mathbb{Z}^{[\frac{m-1}{2}]} \subset \text{Image}([\mathbb{CP}^m, F/O] \xrightarrow{i^*} [\mathbb{CP}^{m-1}, F/O]),$$

and

$$(2.4) \quad [\mathbb{CP}^5, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \text{ and } [\mathbb{CP}^6, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$

Recall that $[X, SF]$ can be identified with the 0-th stable cohomotopy group $\pi^0(M)$.

In [3], Brumfiel also defined an invariant $\sigma : [\mathbb{CP}^m, F/O] \rightarrow \Theta_{2m+1}$ such that the composition

$$[\mathbb{CP}^m, F/O] \xrightarrow{\sigma} \Theta_{2m+1} \xrightarrow{\psi_*} \pi_{2m+1}(F/O) \cong \text{Coker}(J_{2m+1})$$

coincides with the map

$$[\mathbb{CP}^m, F/O] \xrightarrow{p^*} [\mathbb{S}^{2m+1}, F/O] = \text{Coker}(J_{2m+1})$$

induced by the hopf fibration $\mathbb{S}^{2m+1} \rightarrow \mathbb{CP}^m$.

Consider the following long exact sequence associated to the cofiber sequence

$$\mathbb{S}^{2m-1} \xrightarrow{p} \mathbb{CP}^{m-1} \xrightarrow{i} \mathbb{CP}^m \xrightarrow{f_{\mathbb{CP}^m}} \mathbb{S}^{2m} :$$

(2.5)

$$\dots \rightarrow [S\mathbb{CP}^{n-1}, Top/O] \xrightarrow{(S(g))^*} [\mathbb{S}^{2n}, Top/O] \xrightarrow{f_{\mathbb{CP}^n}^*} [\mathbb{CP}^n, Top/O] \xrightarrow{i^*} [\mathbb{CP}^{n-1}, Top/O],$$

where $f_{\mathbb{CP}^m}^* : [\mathbb{S}^{2m}, Top/O] \rightarrow [\mathbb{CP}^m, Top/O]$ is a monomorphism by [10, Theorem 1] for $m \leq 8$. We now prove the following.

Theorem 2.4. (i) $\mathcal{C}(\mathbb{CP}^5) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2$.

(ii) $\mathcal{C}(\mathbb{CP}^6) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2$.

(iii) $\mathcal{C}(\mathbb{CP}^7) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. (i): Since $f_{\mathbb{CP}^4}^* : [\mathbb{S}^8, Top/O] \rightarrow [\mathbb{CP}^4, Top/O]$ is an isomorphism by [9, Theorem 2.3] and $[\mathbb{S}^8, Top/O] \cong \Theta_8 \cong \mathbb{Z}_2$, the non-trivial element in $[\mathbb{CP}^4, Top/O]$ is represented by a map

$$g : \mathbb{CP}^4 \xrightarrow{f_{\mathbb{CP}^4}} \mathbb{S}^8 \xrightarrow{\Sigma} Top/O,$$

where $\Sigma : \mathbb{S}^8 \rightarrow Top/O$ represents the exotic 8-sphere in Θ_8 . Therefore the induced map $p^* : [\mathbb{CP}^4, Top/O] \rightarrow [\mathbb{S}^9, Top/O]$ be such that $p^*(g)$ is represented by the map

$$\mathbb{S}^9 \xrightarrow{p} \mathbb{CP}^4 \xrightarrow{f_{\mathbb{CP}^4}} \mathbb{S}^8 \xrightarrow{\Sigma} Top/O.$$

If the map $f_{\mathbb{CP}^4} \circ p : \mathbb{S}^9 \rightarrow \mathbb{S}^8$ represents the hopf element η in $\pi_1^s \cong \pi_9(\mathbb{S}^8) \cong \mathbb{Z}_2\{\eta\}$, then

$$\eta \circ \eta : \mathbb{S}^9 \xrightarrow{f_{\mathbb{CP}^4} \circ p} \mathbb{S}^8 \xrightarrow{\eta} \mathbb{S}^7$$

is the generator of $\pi_2^s \cong \pi_9(\mathbb{S}^7) \cong \mathbb{Z}_2\{\eta^2\}$. Therefore the composition

$$\mathbb{CP}^4 \xrightarrow{f_{\mathbb{CP}^4}} \mathbb{S}^8 \xrightarrow{\eta} \mathbb{S}^7$$

is not null homotopic, which contradicts the fact that every map $\mathbb{CP}^4 \mapsto \mathbb{S}^7$ is null homotopic. Therefore the map $f_{\mathbb{CP}^4} \circ p : \mathbb{S}^9 \rightarrow \mathbb{S}^8$ is null homotopic. Hence $p^*(g)$ is null homotopic and so $p^* : [\mathbb{CP}^4, Top/O] \rightarrow [\mathbb{S}^9, Top/O]$ is the zero homomorphism. Now by the above exact

sequence (2.5), where $m = 5$, the induced map $i^* : [\mathbb{C}P^5, Top/O] \rightarrow [\mathbb{C}P^4, Top/O]$ is surjective and hence $[\mathbb{C}P^5, Top/O]$ is an abelian group of order 12. Since $[\mathbb{C}P^5, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ ([4]) and by using the fact that the homomorphism

$$\psi_* : [\mathbb{C}P^5, Top/O] \rightarrow [\mathbb{C}P^5, F/O] \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$$

is a monomorphism, we can get

$$\mathcal{C}(\mathbb{C}P^5) \cong [\mathbb{C}P^5, Top/O] \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2.$$

This proves (i).

(ii): If $m = 6$ in the above exact sequence (2.5) and using the fact that $[\mathbb{S}^{12}, Top/O] \cong \Theta_{12} = 0$ and $[\mathbb{S}^{11}, Top/O] \cong \Theta_{11} \cong \mathbb{Z}_{992}$, we have that the induced map

$$i^* : [\mathbb{C}P^6, Top/O] \rightarrow [\mathbb{C}P^5, Top/O]$$

is a monomorphism and the image

$$([\mathbb{C}P^5, Top/O] \xrightarrow{p^*} [\mathbb{S}^{11}, Top/O]) \subseteq \mathbb{Z}_2 \subset \mathbb{Z}_{992}.$$

Since the homomorphism

$$\psi_* : [\mathbb{C}P^6, Top/O] \rightarrow [\mathbb{C}P^6, F/O] \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$$

is a monomorphism and by (i), we have that the homomorphism

$$i^* : [\mathbb{C}P^6, Top/O] \rightarrow [\mathbb{C}P^5, Top/O]$$

is not surjective. Now by the exact sequence (2.5), where $m = 6$,

$$p^* : [\mathbb{C}P^5, Top/O] \rightarrow [\mathbb{S}^{11}, Top/O]$$

is a non-zero homomorphism and hence $[\mathbb{C}P^6, Top/O] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. This proves (ii).

(iii): There is a commutative diagram

$$(2.6) \quad \begin{array}{ccccccc} \mathbb{Z}_2 = [\mathbb{S}^{14}, Top/O] & \xrightarrow{f_{\mathbb{C}P^7}} & [\mathbb{C}P^7, Top/O] & \xrightarrow{i^*} & [\mathbb{C}P^6, Top/O] & \xrightarrow{p^*} & [\mathbb{S}^{13}, Top/O] = \mathbb{Z}_3 \\ \downarrow \psi_* & & \downarrow \psi_* & & \downarrow \psi_* & & \cong \downarrow \psi_* \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 = [\mathbb{S}^{14}, F/O] & \xrightarrow{f_{\mathbb{C}P^7}} & [\mathbb{C}P^7, F/O] & \xrightarrow{i^*} & [\mathbb{C}P^6, F/O] & \xrightarrow{p^*} & [\mathbb{S}^{13}, F/O] \\ \cong \uparrow \phi_* & & \uparrow \phi_* & & \uparrow \phi_* & & \cong \uparrow \phi_* \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 = [\mathbb{S}^{14}, SF] & \xrightarrow{f_{\mathbb{C}P^7}} & [\mathbb{C}P^7, SF] & \xrightarrow{i^*} & [\mathbb{C}P^6, SF] & \xrightarrow{p^*} & [\mathbb{S}^{13}, SF] \end{array}$$

where the rows are part of the long exact sequences obtained from the cofiber sequence

$$\mathbb{S}^{13} \xrightarrow{p} \mathbb{C}P^6 \xrightarrow{i} \mathbb{C}P^7 \xrightarrow{f_{\mathbb{C}P^7}} \mathbb{S}^{14}.$$

Since the image of the homeomorphism $p^* : [\mathbb{C}P^6, SF] \rightarrow [\mathbb{S}^{13}, SF]$ is \mathbb{Z}_3 ([4, Lemma I.9(i)]) and by using the above commutative diagram (2.6), we get that the image of the homeomorphism $p^* : [\mathbb{C}P^6, Top/O] \rightarrow [\mathbb{S}^{13}, Top/O]$ is also \mathbb{Z}_3 . Now by the exactness of the first row in (2.6), we have that $[\mathbb{C}P^7, Top/O]$ is an abelian group of order 4. But $[\mathbb{C}P^7, F/O]_{(2)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by Table 7.5 in [3, pp. 55] and hence $[\mathbb{C}P^7, Top/O] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves (iii). \square

Proposition 2.5. $[\mathbb{C}P^7, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. From the surgery exact sequences of \mathbb{CP}^m and \mathbb{S}^{2m} , we get the following commutative diagram ([5, Lemma 3.4]):

$$(2.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{2m} & \xrightarrow{\psi_* = \eta_{\mathbb{S}^{2m}}} & \pi_{2m}(F/O) & \xrightarrow{\sigma_{\mathbb{S}^{2m}}} & L_{2m}(e) \\ \downarrow = & & \downarrow f_{\mathbb{CP}^m}^* & & \downarrow f_{\mathbb{CP}^m}^* & & \downarrow = \\ 0 & \longrightarrow & \mathcal{S}^{Diff}(\mathbb{CP}^m) & \xrightarrow{\eta_{\mathbb{CP}^m}} & [\mathbb{CP}^m, F/O] & \xrightarrow{\sigma_{\mathbb{CP}^m}} & L_{2m}(e) \end{array}$$

By chasing the diagrams (2.6) and (2.7), where $m = 7$, we have the following facts:

- (1) $\psi_* : [\mathbb{S}^{14}, Top/O] \rightarrow [\mathbb{S}^{14}, F/O]$ is injective.
- (2) $f_{\mathbb{CP}^7}^* \circ \psi_* : [\mathbb{S}^{14}, Top/O] \rightarrow [\mathbb{CP}^7, F/O]$ is injective.
- (3) $\sigma_{\mathbb{S}^{2m}} : [\mathbb{S}^{14}, F/O] \rightarrow L_{14}(e) \cong \mathbb{Z}_2$ is non zero map.
- (4) $f_{\mathbb{CP}^7}^* : [\mathbb{S}^{14}, F/O] \rightarrow [\mathbb{CP}^7, F/O]$ is injective.

These facts together with the exactness of the last row in the diagram (2.6) imply that $[\mathbb{CP}^7, SF]$ is isomorphic to an abelian group of order 8. Again by Table 7.5 in [3, pp. 55], $[\mathbb{CP}^7, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves the proposition. \square

Theorem 2.6. (i) $\mathcal{C}(\mathbb{CP}^8) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
(ii) $[\mathbb{CP}^8, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. (i): Consider the following diagram as in (2.6) for the inclusion $i : \mathbb{CP}^7 \hookrightarrow \mathbb{CP}^8$:

$$(2.8) \quad \begin{array}{ccccccc} \mathbb{Z}_2 = [\mathbb{S}^{16}, Top/O] & \xrightarrow{f_{\mathbb{CP}^8}^*} & [\mathbb{CP}^8, Top/O] & \xrightarrow{i^*} & [\mathbb{CP}^7, Top/O] & \xrightarrow{p^*} & [\mathbb{S}^{15}, Top/O] \\ \downarrow \psi_* & & \downarrow \psi_* & & \downarrow \psi_* & & \downarrow \psi_* \\ \mathbb{Z}_2 \oplus \mathbb{Z} = [\mathbb{S}^{16}, F/O] & \xrightarrow{f_{\mathbb{CP}^8}^*} & [\mathbb{CP}^8, F/O] & \xrightarrow{i^*} & [\mathbb{CP}^7, F/O] & \xrightarrow{p^*} & [\mathbb{S}^{15}, F/O] \cong \text{Coker}(J_{15}) \\ \uparrow \phi_* & & \uparrow \phi_* & & \uparrow \phi_* & & \uparrow \phi_* \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 = [\mathbb{S}^{16}, SF] & \xrightarrow{f_{\mathbb{CP}^8}^*} & [\mathbb{CP}^8, SF] & \xrightarrow{i^*} & [\mathbb{CP}^7, SF] & \xrightarrow{p^*} & [\mathbb{S}^{15}, SF] \end{array}$$

Note that the map

$$\mathbb{S}^{15} \xrightarrow{p} \mathbb{CP}^7 \xrightarrow{f_{\mathbb{CP}^7}} \mathbb{S}^{14} \in \pi_1^s = \mathbb{Z}_2\{\eta\}.$$

Whether it is non-trivial or not is determined by the action of the Steenrod operation Sq^2 on the cone. Since $Sq^2(x^7) = x^8$ and therefore $f_{\mathbb{CP}^7} \circ p : \mathbb{S}^{15} \rightarrow \mathbb{S}^{14}$ represents the element η . Thus the map

$$p^* \circ f_{\mathbb{CP}^7}^* : [\mathbb{S}^{14}, SF] = \pi_{14}^s \rightarrow [\mathbb{CP}^7, SF] \rightarrow [\mathbb{S}^{15}, SF] = \pi_{15}^s$$

is multiplication by η , where

$$\pi_{14}^s = \mathbb{Z}_2\{\sigma^2\} \oplus \mathbb{Z}_2\{\kappa\}$$

and

$$\pi_{15}^s = \mathbb{Z}_{32}\{\rho\} \oplus \mathbb{Z}_2\{\eta \circ \kappa\} \oplus \mathbb{Z}_3\{\alpha_4\} \oplus \mathbb{Z}_5\{\alpha_{2,5}\}$$

[21]. Now by [21, Theorem 14.1 (i), p.190], $\eta \circ \sigma^2 = 0$. Note from the proof of Proposition 2.5 that the map $f_{\mathbb{CP}^7}^* : [\mathbb{S}^{14}, SF] \rightarrow [\mathbb{CP}^7, SF]$ is injective and hence

$$p^* \circ f_{\mathbb{CP}^7}^*(\kappa) = \eta \circ \kappa \neq 0 \in \text{Coker}(J_{15}) \cong \mathbb{Z}_2$$

[19]. Therefore the map

$$\phi_* \circ p^* : [\mathbb{CP}^7, SF] \rightarrow \text{Coker}(J_{15})$$

is non zero homomorphism. Now from the diagram (2.8), we get that the map

$$p^* : [\mathbb{CP}^7, Top/O] \rightarrow [\mathbb{S}^{15}, Top/O]$$

is non zero homomorphism. Since Brumfiel [3] showed that the image of the invariant $\sigma : [\mathbb{CP}^7, SF] \rightarrow \Theta_{15}$ is contained in $\text{Coker}(J_{15}) \cong \mathbb{Z}_2$ and identified $\sigma([\mathbb{CP}^7, SF])$ with the image of the map

$$p^* : [\mathbb{CP}^7, \text{Top}/O] \rightarrow [\mathbb{S}^{15}, \text{Top}/O].$$

Therefore the image of the map

$$p^* : [\mathbb{CP}^7, \text{Top}/O] \rightarrow [\mathbb{S}^{15}, \text{Top}/O]$$

is \mathbb{Z}_2 . Now by the exact sequence (2.5) for $m = 8$, Theorem 2.4(iii) and using the fact that

$$f_{\mathbb{CP}^8}^* : [\mathbb{S}^{16}, \text{Top}/O] \rightarrow [\mathbb{CP}^8, \text{Top}/O]$$

is injective and the image of the map

$$p^* : [\mathbb{CP}^7, \text{Top}/O] \rightarrow [\mathbb{S}^{15}, \text{Top}/O]$$

is \mathbb{Z}_2 , it follows that $[\mathbb{CP}^8, \text{Top}/O]$ is isomorphic to an abelian group of order 4. Again by Table 7.5 in [3, pp. 55], $[\mathbb{CP}^8, \text{Top}/O] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves (i).

(ii): From the proof of Theorem 2.4(iii) and Theorem 2.6(i), we have that

$$[\mathbb{CP}^7, SF] \cong \mathbb{Z}_2\{\sigma^2\} \oplus [\mathbb{CP}^7, \text{Top}/O] \cong \mathbb{Z}_2\{\sigma^2\} \oplus \mathbb{Z}_2\{\kappa\} \oplus \mathbb{Z}_2\{t\},$$

where

$$[\mathbb{CP}^7, \text{Top}/O] = \mathbb{Z}_2\{\kappa\} \oplus \mathbb{Z}_2\{t\}$$

and σ^2 and κ are coming from $[\mathbb{S}^{14}, SF]$ and t is coming from the image of the map

$$i^* : [\mathbb{CP}^8, \text{Top}/O] \rightarrow [\mathbb{CP}^7, \text{Top}/O].$$

This together with the facts that $\eta \circ \sigma^2 = 0$ and $\eta \circ \kappa \neq 0$ as in (i) implies that the image of the map $p^* : [\mathbb{CP}^7, SF] \rightarrow [\mathbb{S}^{15}, SF]$ is $\mathbb{Z}_2\{\eta \circ \kappa\}$ and hence the image of the map

$$i^* : [\mathbb{CP}^8, SF] \rightarrow [\mathbb{CP}^7, SF]$$

is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now by using the diagram (2.8) and the fact that the image of the composition

$$f_{\mathbb{CP}^8}^* \circ \phi_* : [\mathbb{S}^{16}, SF] \rightarrow [\mathbb{CP}^8, F/O]$$

is \mathbb{Z}_2 , we get that the image of the map

$$f_{\mathbb{CP}^8}^* : [\mathbb{S}^{16}, SF] \rightarrow [\mathbb{CP}^8, SF]$$

is \mathbb{Z}_2 . Therefore $[\mathbb{CP}^8, SF]$ is isomorphic to an abelian group of order 8. Again by Table 7.5 in [3, pp. 55], $[\mathbb{CP}^8, SF] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves (ii). □

By Theorem 2.4 and 2.6(i), we have.

Corollary 2.7. (i) *The number of smooth structures on \mathbb{CP}^5 is 12.*

(ii) *The number of smooth structures on \mathbb{CP}^6 is 6.*

(iii) *The number of smooth structures on \mathbb{CP}^7 is 4.*

(iv) *The number of smooth structures on \mathbb{CP}^8 is 4.*

Remark 2.8.

(i) From the proof of Theorem 2.6, we note that the image of the map

$$i^* : [\mathbb{CP}^8, \text{Top}/O] \rightarrow [\mathbb{CP}^7, \text{Top}/O]$$

is \mathbb{Z}_2 .

(ii) Next we consider the map

$$\sigma : [\mathbb{CP}^m, F/O] \rightarrow \Theta_{2m+1}.$$

Recall from [3] that if

$$g \in [\mathbb{CP}^m, Top/O] \subseteq [\mathbb{CP}^m, F/O],$$

then $\sigma(g) \in \Theta_{2m+1}$ is the sphere Σ^{2m+1} admitting free differentiable \mathbb{S}^1 - action such that the orbit space $\Sigma^{2m+1}/\mathbb{S}^1$ is PL-homeomorphic to the complex projective space \mathbb{CP}^m . For $m = 8$, Katsuo Kawakubo [10, Corollary 1] showed that there exists no differentiable free action of \mathbb{S}^1 on an exotic sphere such that the orbit space is PL-homeomorphic to \mathbb{CP}^8 . Therefore the image of the map

$$\sigma : [\mathbb{CP}^8, Top/O] \rightarrow \Theta_{17}$$

is 0. Since the image $\sigma([\mathbb{CP}^8, Top/O]) = p^*([\mathbb{CP}^8, Top/O])$, where

$$p^* : [\mathbb{CP}^8, Top/O] \rightarrow [\mathbb{S}^{17}, Top/O].$$

Hence $p^* : [\mathbb{CP}^8, Top/O] \rightarrow [\mathbb{S}^{17}, Top/O]$ is the zero homomorphism. Now by the exact sequence (2.5) for $m = 9$ and Theorem 2.6, we get that

$$i^* : [\mathbb{CP}^9, Top/O] \rightarrow [\mathbb{CP}^8, Top/O]$$

is surjective.

3. THE SMOOTH TANGENTIAL STRUCTURE SET $\mathcal{S}^{Diff}(\mathbb{CP}^m)$

Definition 3.1. ([2, 17, 18, 16, 22]) Let N and M be closed smooth manifolds. A degree 1 normal map $(N, f, b) : N \rightarrow M$ is a degree one map $f : N \rightarrow M$ together with a stable bundle map $b : \nu_N \rightarrow \xi$ where ν_N is the stable normal bundle of N , b covers f and ξ is some stable vector bundle over M (necessarily fibre homotopy equivalent to ν_M). A normal bordism of degree one normal maps (N_i, f_i, b_i) , $i = 0, 1$ is a degree one normal map $(Z, g, c) : Z \rightarrow M \times [0, 1]$ restricting to (N_i, f_i, b_i) over $M \times \{i\}$. The set of normal bordism classes of degree one normal maps into a closed smooth manifold M is called the normal structure set of M , and we denote it by $\mathcal{N}^{Diff}(M)$.

The normal structure set $\mathcal{N}^{Diff}(M)$ of an m -dimensional smooth closed simply connected manifold M^m fits into the surgery exact sequence of pointed sets

$$(3.1) \quad \cdots \longrightarrow \mathcal{N}^{Diff}(M \times [0, 1]) \xrightarrow{\sigma} L_{m+1}(\mathbb{Z}) \xrightarrow{\theta} \mathcal{S}^{Diff}(M) \xrightarrow{\eta} \mathcal{N}^{Diff}(M) \xrightarrow{\sigma} L_m(e),$$

where the map

$$\eta : \mathcal{S}^{Diff}(M) \rightarrow \mathcal{N}^{Diff}(M)$$

is defined by mapping $[N, f]$ to $[(N, f, b) : N \rightarrow M]$, here $b : \nu_N \rightarrow f^{-1*}(\nu_N)$ is the canonical bundle map and $\xi = f^{-1*}(\nu_N)$, $L_m(e)$ is the surgery obstruction groups, the surgery obstruction maps $\sigma : \mathcal{N}^{Diff}(M) \rightarrow L_m(e)$, $\sigma : \mathcal{N}^{Diff}(M \times [0, 1]) \rightarrow L_{m+1}(e)$ and the action map $\theta : L_{m+1}(e) \rightarrow \mathcal{S}^{Diff}(M)$. Using identity maps as base points we have Sullivans familiar identifications

$$\mathcal{N}^{Diff}(M) = [M, F/O]$$

and

$$\mathcal{N}^{Diff}(M \times [0, 1]) = [SM, F/O],$$

where SM is the suspension of M .

Definition 3.2. [13, 6] Let M be a closed smooth m -manifold with stable normal bundle ν_M of rank $k \gg m$. The smooth tangential structure set of M ,

$$\mathcal{S}^{TDiff}(M) = \{(N, f, b) \mid f : N \rightarrow M, b : \nu_N \rightarrow \nu_M\} / \simeq,$$

consists of equivalence classes of triples (N, f, b) where N is a smooth manifold, $f : N \rightarrow M$ is a homotopy equivalence and $b : \nu_N \rightarrow \nu_M$ is a map of stable bundles. Two structures (N_0, f_0, b_0) and (N_1, f_1, b_1) are equivalent if there is an s -cobordism (U, N_0, N_1, F, B) and where $F : U \rightarrow M$ is a simple homotopy equivalence, $B : \nu_U \rightarrow \nu_M$ is a bundle map and these data restrict to (N_0, f_0, b_0) and (N_1, f_1, b_1) at the boundary of U .

The tangential surgery exact sequence for a simply connected closed smooth manifold M finishes with the following four terms

$$(3.2) \quad L_{m+1}(e) \xrightarrow{\theta} \mathcal{S}^{TDiff}(M) \xrightarrow{\eta^t} \mathcal{N}^{TDiff}(M) \xrightarrow{\sigma} L_m(e),$$

where the definition of $\mathcal{N}^{TDiff}(M)$ is similar to the definition of $\mathcal{S}^{TDiff}(M)$ except that for representatives (N, f, b) we require only that $f : N \rightarrow M$ is a degree one map and the equivalence relation is defined using normal cobordisms over (M, ν_M) . The set $\mathcal{N}^{TDiff}(M)$ of tangential normal invariants of M can be identified with $[M, SF]$.

For $M = \mathbb{CP}^m$, there are exact sequences

$$(3.3) \quad 0 \rightarrow \mathcal{S}^{TDiff}(\mathbb{CP}^{2n+1}) \rightarrow [\mathbb{CP}^{2n+1}, SF] \xrightarrow{\sigma_1} \mathbb{Z}_2 \text{ and } 0 \rightarrow \mathcal{S}^{TDiff}(\mathbb{CP}^{2n}) \rightarrow [\mathbb{CP}^{2n}, SF] \xrightarrow{\sigma_2} \mathbb{Z},$$

where σ_1 is a homomorphism, but σ_2 is not [20].

Proposition 3.3. *The surgery obstruction $\sigma_1 : [\mathbb{CP}^7, SF] \rightarrow \mathbb{Z}_2$ is non zero map.*

Proof. By using the digram (2.7) for $m = 7$ and the fact that the map

$$f_{\mathbb{CP}^7}^* : [\mathbb{S}^{14}, F/O] \rightarrow [\mathbb{CP}^7, F/O] = \mathbb{Z}^3 \oplus [\mathbb{CP}^7, SF]$$

is injective, we get that the surgery obstruction $\sigma_1 : [\mathbb{CP}^7, SF] \rightarrow \mathbb{Z}_2$ is non zero map. This proves proposition. \square

Remark 3.4. By [4, Lemma I.5], the surgery obstruction $\sigma_2 : [\mathbb{CP}^8, SF] \rightarrow \mathbb{Z}$ is the zero map.

The following result follows from the exact sequence (3.3) and Proposition 3.3, Remark 3.4, Proposition 2.5 and Theorem 2.6.

Theorem 3.5. (i) $\mathcal{S}^{TDiff}(\mathbb{CP}^7) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(ii) $\mathcal{S}^{TDiff}(\mathbb{CP}^8) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

By the surgery exact sequence (3.1), the set of homotopy smoothings of \mathbb{CP}^m , $\mathcal{S}^{Diff}(\mathbb{CP}^m)$, embeds in $[\mathbb{CP}^m, F/O]$. Also there is an exact sequence

$$\pi_s^0(\mathbb{CP}^m) = [\mathbb{CP}^m, SF] \hookrightarrow [\mathbb{CP}^m, F/O] \rightarrow \widetilde{KO}(\mathbb{CP}^m).$$

It follows that $\pi_s^0(\mathbb{CP}^m)$ is the torsion subgroup of $[\mathbb{CP}^m, F/O]$. If the homotopy equivalence $f : M \rightarrow \mathbb{CP}^m$ represents an element of $\mathcal{S}^{Diff}(\mathbb{CP}^m)$, its image in $\widetilde{KO}(\mathbb{CP}^m)$ is given by $(f^{-1})^*TM - T\mathbb{CP}^m$. Since the image of $[\mathbb{CP}^m, F/O]$ is contained in the free part of $\widetilde{KO}(\mathbb{CP}^m)$ and is determined by the Pontryagin classes of M ([1, Lemma 2.25]). In fact, the Pontryagin character $ph = ch \circ c$, c is injective except for 2-torsion, and ch is injective since $H^*(\mathbb{CP}^m, \mathbb{Z})$ is torsion free. Therefore we have the following results:

Corollary 3.6. *The smooth structures on \mathbb{CP}^m having the same Pontryagin classes of \mathbb{CP}^m are in 1-1 correspondence with the elements of the kernel of the map*

$$\sigma : [\mathbb{CP}^m, SF] \rightarrow L_{2m}(e).$$

The computations of the groups $\pi_s^0(\mathbb{CP}^m) = [\mathbb{CP}^m, SF]$ and of the surgery obstruction $\sigma : [\mathbb{CP}^m, F/O] \rightarrow L_{2m}(e)$ have been made by Brumfiel [3, 4] for $m \leq 6$. Now by Proposition 3.3, Remark 3.4, Proposition 2.5 and Theorem 2.6, we get:

Theorem 3.7. (i) *There are exactly four smooth structures on \mathbb{CP}^7 having the same Pontryagin classes of \mathbb{CP}^7 .*
(ii) *There are exactly eight smooth structures on \mathbb{CP}^8 having the same Pontryagin classes of \mathbb{CP}^8 .*

REFERENCES

- [1] W. Browder, Surgery and the theory of differentiable transformation groups, *Proc. of the Tulane Symp. on Transformation Groups*, pp. 1-46, Springer Verlag, Berlin (1968).
- [2] W. Browder, Surgery on simply-connected manifolds, *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65. Springer-Verlag*, New York - Heidelberg (1972).
- [3] G. Brumfiel, Differentiable \mathbb{S}^1 - Actions on Homotopy Spheres, mimeographed. University of California, Berkeley (1968).
- [4] G. Brumfiel, Homotopy equivalences of almost smooth manifolds, *Comm. Math. Helv.*, **46** (1971) 381-407.
- [5] D. Crowley: The smooth structure set of $\mathbb{S}^p \times \mathbb{S}^q$, *Geom. Dedicata*. **148**, (2010) 15-33.
- [6] D. Crowley and I. Hambleton, Finite group actions on Kervaire manifolds, *Advances in Mathematics*(to appear) 2015.
- [7] W.-C. Hsiang, A note on free differentiable actions of \mathbb{S}^1 and \mathbb{S}^3 on homotopy spheres, *Ann. of Math.*, (2) **83** (1966), 266-272.
- [8] W.-C. Hsiang and W.-Y. Hsiang, Some free differentiable sections of \mathbb{S}^1 and \mathbb{S}^3 on 11-spheres, *Quart. J. Math. Oxford Ser.*, (2) **15** (1964), 371-374.
- [9] R. Kasilingam, The Classification of Smooth Structures on a Homotopy Complex Projective Space, *Proc. Indian Acad. Sci. (Math. Sci.)* Vol. **126**, No. 2, May 2016, pp. 277-281.
- [10] K. Kawakubo, Inertia groups of low dimensional complex projective spaces and some free differentiable actions on spheres I, *Proc. Japan Acad.* **44** (1968), 873-875.
- [11] M. A. Kervaire, J. W. Milnor, *Groups of homotopy spheres I*, *Ann. of Math.* (2) **77** (1963), 504-537.
- [12] R. C. Kirby, L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, *Annals of Mathematics Studies* **88** (1977).
- [13] I. Madsen, L. R. Taylor, and B. Williams, Tangential homotopy equivalences, *Comment. Math. Helv.*, **55** (1980), 445-484.
- [14] D. Montgomery and C. T. Yang, Differentiable actions on homotopy seven spheres, *Trans. Amer. Math. Soc.*, **122** (1966), 480-498.
- [15] D. Montgomery and C. T. Yang, Free differentiable actions on homotopy spheres, *Proc. Conf. on formation Groups* (New Orleans, La., 1967), *Springer, New York*, 1968, pp. 175-192.
- [16] W. Luck, A basic introduction to surgery theory, *ICTP Lecture Notes Series 9, Band 1, of the school High-dimensional manifold theory in Trieste*, May/June 2001, Abdus Salam International Centre for Theoretical Physics, Trieste (2002).
- [17] A. A. Ranicki, Algebraic L -theory and topological manifolds, *Cambridge Tracts in Math.*, vol. **102**, Cambridge Univ. Press, Cambridge (1992).
- [18] A. A. Ranicki, Algebraic and geometric surgery, *Oxford Mathematical Monographs*, Oxford University Press, (2002).
- [19] D. Ravenel, Complex cobordism and stable homotopy groups of spheres, *AMS Chelsea Publishing*, Volume **347**, 2003.
- [20] D. Sullivan, Triangulating and smoothing homotopy equivalences and homeomorphisms, in: *The Hauptvermutung Book. A Collection of Papers of the Topology of Manifolds, K-Monogr. Math. 1*, Kluwer Academic Publishers, Dordrecht (1996), 69-103.
- [21] H. Toda, Composition methods in homotopy groups of spheres, *Annals of Mathematics Studies*, No. **49**, Princeton University Press, Princeton, N.J. 1962.
- [22] C. T. C. Wall, Surgery on compact manifolds, Second edition. Edited and with a foreword by A. A. Ranicki. *Mathematical Surveys and Monographs*, **69**. American Mathematical Society, Providence, RI, (1999).

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